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PYTHAGORAS

THE MATHEMATICS TEACHER

Vol. XXIII No. 3

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Some Difficulties Involved in Solving Verbal Problems in Elementary Algebra

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WHAT DIFFICULTIES do pupils meet in solving verbal problems in elementary algebra? Can these difficulties be classified or analyzed? Do these difficulties indicate specific weaknesses in teaching? Do they provide suggestions for remedial teaching? The study here reported was conducted in order to answer these questions in terms of certain typical classroom situations.

The investigation involved 80 pupils in four ninth-year elementary algebra classes in three high schools of Syracuse, New York: one class of 25 pupils in North High School, one class of 19 pupils in Central High School, two classes of 18 pupils each at Roosevelt Junior High School. Each pupil solved, showing all work, seven verbal problems. The problems were administered near the end of one year of elementary algebra, 1929. Two regular periods of forty minutes each were used for the examination. The seven problems were selected from a composite list compiled from problems which experienced teachers found normally gave their pupils difficulty. Some may claim that these problems are not "good" ones, that they are not

real and practical. Others may say that they are too difficult as shown by the results. It may reasonably be claimed that not more than one distance problem, or mixture problem, or percentage problem would occur normally in the same examination. These problems do represent, however, types that are in general use in schools at the present time. Problem 1 served as a psychological "shock absorber" and also provided a wider distribution, due to its lack of difficulty. The seven problems upon which the investigation was based, follow:

1. Admission to a radio concert was \$0.50 for adults and \$0.35 for children. The proceeds from 250 tickets were \$117.50. How many tickets of each kind were sold?
2. A train running 40 miles an hour left a station 45 minutes before a train running 45 miles an hour. In how many hours will the second train overtake the first?
3. Two quarts of alcohol were mixed with five quarts of water; find the number of quarts of alcohol that must be added to make the mixture three-fourths alcohol.
4. A and B, partners in a business, agree to divide their profits so that A will have 30% more than B, since A owns the building. How should they divide a profit of \$18,400?
5. A country grocer has two boxes of tea, one kind worth 60 cents a pound and the other worth 90 cents a pound. He is continually troubled by the question, "What's the difference?"; so he decides to mix the tea and have one price. He wants to fill a 120 pound case and sell it for 80 cents a pound without cheating himself or anyone else. He does not know how much of each kind to use. Jim, a high school boy, solves the problem by algebra. How many pounds of each kind did Jim find should be used?
6. A man walks at the rate of 4 miles an hour. How far can he walk into the country and ride back on a trolley that travels at the rate of 20 miles an hour, if he must be back home 3 hours from the time he started?
7. A merchant sold a suit of clothes and a pair of shoes for \$75, making a profit of \$19.50 on both. His profit on clothing was 25% of the selling price; on shoes it was 30% of the selling price. What did he pay for the suit? For the shoes?

Interpretation of Data

I. Relative Difficulty of Problems

Table I shows the number of pupils in each class who attempted each problem, and the number who solved each problem correctly. For example, in Class 1, 25 pupils attempted problem 1, and 23 solved problem 1 correctly. It was considered that a pupil had attempted

a problem if he had done any algebraic work at all relative to it. It was not considered that a pupil had attempted a problem if he merely put down a few arithmetical computations. At the right is a summarized statement of this group of cases not regarded as "attempts."

Number of Problem	Number of Pupils
1	4
2	2
3	4
4	7
5	0
6	5
7	9

TABLE I

Showing Distribution of "attempts" and "corrects"

No. of Problem	Class 1		Class 2		Class 3		Class 4		Total	
	A	B	A	B	A	B	A	B	A	B
1	25	23	19	18	17	8	14	4	75	53
2	22	4	11	0	15	0	15	0	63	4
3	24	6	15	3	12	0	11	0	62	9
4	18	0	12	1	15	0	13	0	58	1
5	20	11	13	7	18	3	15	8	66	29
6	25	22	8	0	14	0	13	2	60	24
7	23	4	12	1	15	1	12	0	62	6

A. No. of pupils attempting a specific problem.

B. No. of pupils solving a specific problem correctly.

Class 1.....25 pupils Class 3.....18 pupils

Class 2.....19 pupils Class 4.....18 pupils

TABLE II

Showing a Second Distribution of "attempts" and "corrects"

No. of Problems	Class 1		Class 2		Class 3		Class 4		Total	
	A	B	A	B	A	B	A	B	A	B
7	14	0	4	0	6	0	6	0	30	0
6	4	1	2	0	8	0	5	0	19	1
5	5	2	4	0	1	0	3	0	13	2
4	2	2	4	0	1	0	1	0	8	2
3	0	9	3	4	1	1	0	1	4	15
2	0	9	2	4	0	2	0	2	2	17
1	0	1	0	10	1	5	2	7	3	23
0	0	1	0	1	0	10	1	8	1	20

A—No. of pupils attempting a definite No. of Problems.

B—No. of pupils solving correctly a definite No. of problems.

Table II shows for each class the number of pupils attempting, and the number of pupils solving correctly a definite number of problems. For example, in class 1, 4 pupils attempted 6 of the problems and 1 pupil solved 6 of the problems correctly. Table II shows that 20 of the 80 pupils did not solve a single problem correctly; also that no pupil solved all the seven problems correctly. Tables I and II taken together show the relative distribution of effort by the pupils in the four classes and on the seven problems. They show the degree of difficulty of the various problems in terms of the several classes.

II. Specific Difficulties

A detailed study of the eighty papers was made, and the errors examined and classified. Space permits only a summarized statement of this detailed analysis, with inferences:

.1 Problem solving depends on problem reading.

Many of the errors on the papers indicated directly or indirectly inability to read understandingly the problem. A careless disregard of prepositions and other punctuation marks is responsible for many errors in problem reading. A good silent reading test is one of the best instruments for discovering pupils' difficulties in algebra. Pupils must understand the problem so well that they can translate it into algebraic symbols if they are to solve verbal problems effectively.

2. Inability to do logical reasoning, as would be expected, proved to be a significant factor in producing errors.

Pupils seemed to be more efficient in the mechanical aspects of the work than in the reasoning phases. This study would seem to justify Richards' statement: "The majority of pupils who fail to pass algebra are fully able to master the fundamental operations that involve merely a mechanical application of formal rules, but fail because they cannot perform the reasoning requisite for the solution of statement problems."[1] Algebra demands in the field of verbal solution that the pupil get the problem clearly in mind, suspend judgment, scrutinize the available data, and choose pertinent data for his operation. Too much emphasis has been given to memory work and too little to problem "thinking." In the distance problems, many pupils failed to recognize what factors were given. They vaguely knew that three

[1] Richards, H. F., "The Teaching of Algebra," *The Mathematics Teacher*, 16:41.

elements were concerned: distance, rate, and time. But they could not decide which were given and which were to be found. They were uncertain of relationships existing among the elements. Many wrote d = *rt* regardless of what they were given; some had $\frac{\text{rate}}{\text{time}} =$ dist. or $\frac{\text{rate}}{\text{dist.}} =$ time or $\frac{\text{time}}{\text{rate}} =$ dist. If a teacher asks a pupil in an algebra class: "If you drive at the rate of 35 miles an hour, how long will it take you to go to (naming some town to which he knows the distance), he will invariably give the correct answer. But give the pupil the same thing in an algebra problem, and he has no idea how to get it. Ligda seems to be right when he says: "Present day methods do not develop the ability to solve problems; they merely teach problems."[2]

3. Many capable pupils fail to master the solution of problems because they are not taught to go at it systematically.

In this experimental study it was found that lack of systematic treatment, lack of proper labeling, and lack of proper statements, seemed to be ultimately responsible for most of the errors. Some of the pupils seemed to have no idea how to begin the solutions of the problems. Thorndike says[3] that no pupil in algebra lets x equal the horses; that he lets x equal the number of horses, and proceeds to drop the idea of horses out of his consideration. However, in actual practice, this does not seem to be true. Thorndike was not speaking of the poorer pupils. Pupils do indeed let x equal almost anything. Most of the difficulty in the solution of problems would be eliminated if pupils were required to state properly what their unknown letter or letters are to represent, and to follow some system in arranging their work, being sure that they give all statements necessary for obtaining the equation. Very few pupils can reason sufficiently well to write the required equation or equations without these preliminary steps. After the equation has been formed, the majority of pupils have little or no difficulty in solving it. There were 15 cases on the papers studied where the pupil did not even tell what his unknown quantities were supposed to represent, but wrote the

[2] Ligda, Paul, *Systematic Analysis and Solution of Quantitative Problems*, School Science & Mathematics, 26:177.

[3] Thorndike, E. L., *The Psychology of Arithmetic*, 86.

equations as the first steps in the solutions. It is interesting to know that the following numbers of errors can be traced directly or indirectly to lack of proper labeling on each of the seven problems:

Number of Problem	Number of Errors
1	28
2	6
3	29
4	16
5	32
6	9
7	67
Total	187

Many of these errors would have been discovered by the pupils if they had used the necessary statements for forming their equations. Pupils must be taught the importance of statements in the formation of the equations. The attempt that has sometimes been made, however, to reduce the solution of all problems to mere mechanical application of formulas does not seem to secure the best results. Pupils not only forget the correct statement of the formula but also are not able to apply it if the problem varies somewhat from the type upon which they have been drilled.

4. Lack of knowledge or the application of knowledge of arithmetic, was responsible for many errors.

Decimal points seemed to appear and disappear at the whim of the pupil. Some pupils attempted to do part or all of the problem by arithmetic. In such cases the work usually consisted of a little addition here, a little multiplication there, and a little division somewhere else, ending up with a number labeled, "Ans." There was no system used and the work was meaningless in these cases.

5. The lack of proper checking proved to be a significant cause of failure in solving the problems.

In the study very few papers showed any attempt on the part of the pupils to check their own work. The few pupils who did attempt to check merely substituted their answers into the equations which they had formed with no regard for the original problems. Pupils must be made to realize that such a procedure is not checking a problem, that if they have correctly solved the equation which they have formed, the answers will check whether they are correct or not. If the equation is correct, the answers will be correct, but if the equation is incorrect the answers will still check but will be

incorrect. To check a problem it is necessary for us to make sure that all answers fulfill all the conditions of the problems concerned.

It is not surprising that pupils cannot tell whether their problems check or not when one notes the number of problems whose answers bear no labels. On the papers examined, the following number had answers which bore no labels. (See diagram at right.) Not only do pupils fail to check problems properly but they fail to question illogical and impossible answers. Such answers as "$x = 56\ \frac{3}{8}$ adults," "$y = 193\ \frac{3}{8}$ children," appeared on the papers.

Number of Problem	Number of Papers
1	11
2	2
3	0
4	2
5	6
6	5
7	5
Total	31

Summarized Conclusions and Suggestions

1. The general opinion that problem solving in algebra causes more difficulty than mechanical manipulation, seems justified. The common statement of pupils, "I just never could do problems," is verified in terms of this experiment. This study indicates that the weakest part of classroom instruction is the teaching of verbal problems.
2. Special emphasis should be given to the teaching of problem solving in elementary algebra. Instruction should involve specific techniques. Teachers should recognize that when properly taught verbal problems constitute an excellent means of reviewing mechanical processes.
3. A significant ability demanded in problem solving is the ability to think things through.
4. Textbooks and teaching aids should give more assistance to teachers in teaching verbal problems. Teacher training courses in mathematics should give more specific attention to problem solving.
5. The most common causes of failure in solving verbal problems are: lack of preparation and knowledge of techniques on the part of the teacher; on the part of the pupils,—inability to read, lack of logical reasoning ability, poor labeling, lack of statements, lack of knowledge of arithmetic, and lack of proper checking.

The Value of Pupils' Examination Papers to the Teacher

By Joseph B. Orleans, *Chairman, Mathematics Department, George Washington High School, New York City*

For many years teachers have been composing examinations in algebra and in geometry and pupils have been writing the answers to the questions. For many years also the teachers have been permitting the pupils to look over the corrected papers and have been explaining to the class as a whole the errors made by the individuals, while the pupils have been trying to find out why the mark recorded in the upper left hand corner in red or in blue is not more than it happens to be. The teachers have religiously entered the marks in their rating books, in order to consult them most conscientiously in determining the numerical evaluation of the pupils' achievement at the end of the set marking periods in accordance with the regulations of the schools. Individual teachers have at times, no doubt, introduced innovations into their school routine and have tried to learn as much from the examinations as possible; but, in general, from the point of view both of the teacher and of the pupil the examinations have been useless gestures, i.e., as far as the use of the written papers is concerned as an integral part of the classroom work.

What is the purpose of an examination? At the end of a semester or of a year, just before promotion, an achievement test will tell the teacher whether or not the pupil has mastered enough of the work taught to enable him to go on with the work of the next grade. At other times the examination should serve more for the purpose of diagnosis than for anything else. Each test should be followed by remedial teaching. If the papers show a general lack of understanding of a topic, the teacher must teach it again. In order to provide for mistakes that occur on individual papers, the teacher should prepare individual assignments and follow them up until she is certain that the errors have been fully corrected. This is more easily said than done. It is difficult enough to do even with small classes; and when one considers the teaching programs in the large city high schools with two or more preparations and with five classes containing

a total of from 150 to 200 and more pupils, one feels that the preparation of such remedial measures becomes an impossible task. What then can a teacher do to make full use of the examinations as they are given at intervals throughout the year? Or better still, what can the supervisor, who supposedly has more time for such things, do to help the teacher in this phase of the work?

This article attempts to answer these questions in describing what is being attempted in one school. The papers written in a series of tests during the past year were summarized under the supervision of the chairman of the department and the tabulated results were submitted to the teachers. This is only a beginning of what should become a major activity in every department.

The following is a summary of the number of pupils who answered each question right and the number wrong in several tests in algebra. Here only the total for the department is given. In the sheets submitted to the teachers the results for the individuals were included.

FIRST TEST

	Question	*Right*	*Wrong*
1.	Express algebraically		
	(a) the sum of a and b.	370	23
	(b) Seven times the product of c and d.	307	86
	(c) Eight more than y.	337	56
	(d) a increased by b.	359	34
2.	If $c = 5$, the value of $3c^2$ is	282	111
3.	If $d = 4$, the value of $\frac{1}{8}d^4$ is	245	148
4.	If $l = 10$ and $w = 6$, the value of $2l + 3w$ is	354	39
5.	$-9p + 11p =$	256	137
6.	$8x^3 - 14x^3 =$	208	185
7.	$9ab - 10ab - 13ab + 7ab =$	181	212
8.	In $A = hb$, if $h = 6$ and $b = 14$, $A =$	349	44
9.	In $A = bh$, if $A = 90$ and $h = 10$, $b =$	239	154
10.	In $I = prt$, if $p = \$500$, $r = .06$, $t = 3$, $I =$	303	90
11.	If $7x = 21$, $x =$	374	19
12.	If $15 = 3w$, $w =$	323	70
13.	If $3y = 7$, $y =$	288	105
14.	If $9x - 3x = 72$, $x =$	349	44
15.	If the sum of two numbers is 18 and one of them is s, the other one is	208	185
16.	The total cost of n pounds at x cents a pound and m pounds at y cents a pounds is	218	175
17.	Each of n bags of grain weighs k pounds. Let w be the total weight. Express in a formula the relation between w, n and k.	324	69

SECOND TEST

1.	In $A = tw$, if $t = 16$ and $w = 7$, $A =$	336	63
2.	In $A = \frac{1}{2}bh$, if $b = 2\frac{1}{2}$ and $h = 6$, $A =$	310	89
3.	In $A = \pi r^2$, if $\pi = 22/7$ and $r = 7$, $A =$	290	109
4.	The number of cents in d dimes and c cents is	209	190
5.	A boy has n marbles and loses a of them. The number left is	342	57
6.	A man is x years old. In 3 years his age will be	287	112
7.	An auto travels m miles per hour. In 6 hours it will go ..... miles.	369	34
8.	If the yearly rent is d dollars, the monthly rent is	252	147
9.	If the difference between two numbers is d and the smaller one is 8, the larger one is	220	179
10.	The length of a rectangle is T and the width is three feet less than the length. The width is	247	152
11.	The sum of n and another number is 24. The other number is	203	196
12.	If $5a = 35$, $a =$	345	54
13.	If $3x = 2$, $x =$	240	159
14.	If $10a = 5$, $a =$	262	137
15.	If $9a - 6a = 12$, $a =$	318	81
16.	If $a = 2$ and $b = 3$, the value of $a^3 + b^3$ is	215	184
17.	If $c = 5$ and $d = 4$, the value of $3c^3d$ is	252	147
18.	$7z - 3z + 5z - 2z =$	353	46
19.	The formula for s (selling price) in terms of g (gain) and c (cost) is	353	46
20.	Four times the square of c is represented by	211	188

THIRD TEST

1.	$15a^3 - 7 - 3a - 16a^3 - 20a^2 - 10a^2 + 9a + 4 =$	219	174
2.	From $x^3 - 5x^2 + 6x$ take $2x^2 - 3x$.	233	160
3.	$2x \cdot 3x =$	312	81
4.	$5a^3 \cdot 4a^3 =$	327	66
5.	If $x = 3$, and $y = 2$, the value of $x^2 - 9xy + 4y^2$ is	136	257
6.	$2x \cdot 5y =$	326	67
7.	$2n(-4n) =$	290	103
8.	$(-3n^3)\ (4n) =$	272	121
9.	$-3(7a^3) =$	290	103
10.	$a(a + 1) =$	175	218
11.	$-a^3(a + b) =$	162	231
12.	$4x(2 - x) =$	216	177
13.	$5x^3(x^3 - 2) =$	220	173
14.	$-a^3(a^3 + 3) =$	170	223
15.	$-x(-y)(-3z) =$	152	241
16.	$(y - 1)(y + 2) =$	104	289
17.	If $4x - 3 = 3x + 4$, $x =$	231	162
18.	If $8y - 6 = -1 - 12y$, $y =$	161	232

19.	In $F = \frac{9}{5}C + 32$, if $C = 90$, $F =$	226	167
20.	In $F = mv^2/gt$, if $m = 175$, $v = 25$, $g = 32$, $t = 5$, $F =$	149	244

FOURTH TEST

1.	Simplify $a/2 - b/4$	403	81
2.	Simplify $a/x + b/2x$	394	90
3.	Simplify $1/x - 1/(x+1)$	272	212
4.	Simplify $x/(x-y) - y/(x+y)$	320	164
5.	$a/(b-a) = -?/?$	396	88
6.	$y/(x+y) = ?/(y+x)$	295	189
7.	Simplify $6/(a-2) + 2/(2-a)$	390	94
8.	$x + 5/(x+3) - 4/(3-x)$	354	130
9.	Solve for x and y: $0.5x + 0.4y = 5.8$, $0.3x + 0.7y = 6.7$	419	65
10.	If $T = 1/(a+t)$, $a =$	347	137

FIFTH TEST

1.	Solve $x/5 + x/2 = 30$	307	170
2.	Solve $5x/2 - 5x/4 = 9/4 - (3-x)/2$	374	103
3.	Solve $\frac{2x-1}{2x+1} = \frac{3x-2}{3x}$	297	180
4.	Solve $\frac{4}{x+1} - \frac{x+1}{x-1} = \frac{3-x^2}{x^2-1}$	91	386
5.	Solve for x and y: $x + y = 11$, $x - y = 7$	322	155
6.	Solve for x and y: $2x + 3y = 13$, $5x - 4y = 21$	244	233
7.	$x/5 - y/10 = 3/4$, $x/3 + y/5 = 13/6$	284	193
8.	Solve $2x/5 - 3 = x/15 + 1$	358	119

SIXTH TEST

1.	If $ax = b + c$, $x =$	471	20
2.	If $a = b - cx$, $x =$	338	153
3.	If $ax + b^2 = a^2 - bx$, $x =$	376	115
4.	Solve for y: $ax + by = c$, $bx + ay = d$	375	116
5.	Solve for x: $ax + y = m$, $bx - y = n$	371	120
6.	Solve for f: $1/f = 1/p + 1/q$	345	146
7.	In $A = \frac{1}{2}h(a+b)$, if $a = 18.5$, $b = 8.5$, $h = 10$, $A =$	381	110

In studying the above summaries various questions arise. Why did only 23 pupils out of 393 fail to express the sum of a and b correctly, while 86 failed with seven times the product of c and d? Does the trouble lie in the meaning of the word product? Why did 111 fail to evaluate $3c^2$ correctly? What is the difference between the solution of $7x = 21$ and that of $3y = 7$ which caused only 19 incorrect answers for the former, but 105 for the latter? Why did 160 out of 393 fail with a simple subtraction example which required taking $2x^2 - 3x$ from $x^3 - 5x^2 + 6x$? Why did 170 out of 477 fail with an easy equation like $x/5 + x/2 = 30$? What difficulty in the pair of equations $x + y = 11$ and $x - y = 7$ prevented 155 out of 477 from getting the correct answer? A teacher who compares the showing of her group with that of the entire department is able to locate in a general way points of weakness in her teaching that need special attention. If she supplements this with a study of the summary of typical errors, she has a good foundation on which to base changes in her methods of presenting and developing the various topics.

The following is a list of errors found in groups of test papers in algebra:

(1) Solve for x: $ax = b + c$ — Correct answer 471; Wrong answer 20

(The numbers in parentheses indicate how many pupils wrote each of the wrong answers.)

$\frac{b+c}{-a}$ ..... (5) $\frac{a}{b+c}$ ..... (7)

$\frac{a+b}{c}$ ..... (2) $a(b+c)$ ..... (2)

$\frac{b}{a+c}$ ..... (1) an answer containing x ..... (3)

(2) Solve for x: $a = b - cx$ — Correct answer 338; Wrong answer 153

$\frac{a}{b-c}$ ..... (55) $\frac{a-b}{c}$ ..... (40) $\frac{b-c}{a}$ ..... (26)

$\frac{b}{a+c}$ ..... (5) $\frac{b-a}{-/c}$ ..... (4) $\frac{b+c}{a}$ ..... (3)

$\frac{a+b}{c}$.....(3) $\frac{b-c}{-a}$.....(3) $c+a-b$.....(5)

(3) Solve for x: $ax+b=a-bx$. Correct answer 376
Wrong answer 115

Error in transposition.....(16)
Answer not factored.....(49)
Error in factoring.....(15)
Error in reduction in answer.....(5)
Error in interpretation.....(10)

(4) Solve for y: Correct answer 375
$bx+ay=d$ Wrong answer 116

Incorrect elimination.....(38)
Error in getting like terms.....(21)
Incorrect factoring.....(13)
Incorrect division in last step.....(5)
Wrong interpretation.....(8)

(5) Solve for x: Correct answer 371
$bx-y=n$ Wrong answer 120

Error in elimination.....(66)
$m+n$ written as mn.....(14)
$ax+bx$ factored as $2x\ (a+b)$.....(1)
Incorrect division.....(3)

(6) Solve for q: $1/f=1/p+1/q$ Correct answer 345
Wrong answer 146

Error in multiplying by LCD.....(48)
Error in transposing.....(42)
$fq+fp$ factored as $2f\ (q+p)$.....(3)
$fq+fp$ factored as $2f\ (qp)$.....(10)

(7) Solve $x/2+x/3=30$ Correct answer 307
Wrong answer 170

Neglected to multiply 30 by 6.....(147)
Error in multiplication.....(10)
Error in division.....(7)
Error in combining terms.....(6)

(8) Solve $5x/2-5x/4=9/4-\frac{3-x}{2}$ Correct answer 374
Wrong answer 103

Error involving minus sign before the last fraction (35)
Error in combining terms.....(20)
Wrong LCD.....(6)
Eddor in division.....(6)
$2\ (3-x)$ written as $6-x$.....(10)

A summary of typical errors like this one offers unlimited opportunities for the improvement of teaching. If a teacher knows that there is a tendency on the part of the pupils to make certain mistakes, it is inconceivable that with the proper preparation he should not bring about a better understanding of the topic that is being taught. This is of especial value to young and inexperienced teachers. It is very fine to let a teacher work out her own salvation and to learn by experience; but if this means experimenting with the children, then it is far better to give the beginner the experience of others in crystalized form. An important part of a course in methods of teaching algebra and geometry should be the study of typical errors actually made by children on papers that they have written.

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Individual Work in Plane Geometry

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IN THE OCTOBER 1929 number of the MATHEMATICS TEACHER there appeared a report of individual work done in ninth grade algebra.[1] The following report may be of interest as a description of an attempt to teach plane geometry by an individual instruction method. It was used in the Russell High School of the Southeastern State Teachers College, Durant, Oklahoma during the winter and spring of 1927-28. The class used was composed of about twenty-five members which were all the pupils taking geometry in this particular school. The text book in use was the state adopted text of Oklahoma, Newell and Harper's Plane Geometry, published by Row-Peterson and Company.

THE PLAN

The plan selected was to give to each pupil a work sheet with the assignments for a three-week or possibly a four-week period. The idea of the work sheet and the general method of procedure were borrowed from the Ardmore High School of Ardmore, Oklahoma. A sample of the work sheet is reproduced on the next page.

This work sheet gave the pupil his assignment for the three-week period. It was designed so that all pupils covered the "minimum essentials," and was intended to include enough to tax the capacity of the best. The starred problems, required of everybody, were selected by the writer as those with which the pupil would have to be familiar in order to go on with the course. The standards set for the marks A, B, C, D, and F were purely arbitrary and were sometimes found to be too low and on other sheets placed too high. The class was taught by the writer and a diary of impressions was kept for the entire period.

HOW THE WORK SHEET WAS USED

The pupils were told to take their books and to begin on the assigned problems. During the class period the pupils could get help

[1] Martha C. Cooke, "Individual Work in Algebra." *Mathematics Teacher*, 22:261-365.

STUDENTS WORK SHEET I

Plane Geometry First three weeks of the second semester

Students Name_____

The Assignments marked * are required of all students, as well as the definitions listed below. Every pupil is also expected to memorize all theorems he proves. Extra credit will be given for work above the requirement. The numbers to the left in each pair of numbers refer to the propositions, articles, or exercises, while the numbers to the right refer to the pages in the book.

	Page	F	D	C	B	A		Page	F	D	C	B	A
*I	130						Ex. 12	139					
*Cor. I	132						Ex. 14, 15	139					
Ex. 1, 2	132						Ex. 16	140					
*II	132						Ex. 20	140					
*Ex. 2, 4	133						IX	141					
Ex. 5	133						Ex. 2	142					
*III	133						X	142					
*Cor.	133						Cor.	143					
Ex. 1	133						Ex. 2, 3	143					
IV	134						XI	144					
*V	135						Cor.	145					
*Cor. I	135						Ex. 2	145					
*Cor. II	136						Ex. 3	145					
*VI	137						XII	145					
*VII	137						XIII	146					
Ex. 1, 2	137						Ex. 1, 2	146					
VIII	138						Ex. 3	146					
*Ex. 1, 2	139						Ex. 4	146					
*Ex. 3, 4	139						Ex. 5	146					
Ex. 5	139						XIV	147					
Ex. 6	139						Ex. 1, 2	147					
Ex. 7	139						Ex. 3	147					
Ex. 9	139						Ex. 4	148					
Ex. 11	139						Ex. 6	148					

Definitions to be learned: Proportionality (p. 132) Similar polygons (p.134) Division of a line externally and internally (p. 140).

STANDARDS

	D	C	B	A
1st Week	8	10	12	15
2nd Week	16	20	24	30
3rd Week	24	30	36	45

from the teacher or from any of their fellows. When they felt that they knew the proof of a proposition, corollary or exercise they were instructed to bring it to the teacher or to a monitor and explain it, using their own figure rather than the one in the book. The monitors were the extra good pupils in the class, and were appointed by the teacher from his knowledge of their ability, gained during the early part of the geometry course. There were five in this particular group. The monitors were asked not to hear any pupil on a problem on which they themselves had not been checked. Upon the completion of the pupils explanation he was given a mark by checking in the appropriate column, and the person who heard him initialed the paper and also checked the mark on the instructor's sheet. In this way both the pupil and the instructor knew where they stood at any time. The class room became a busy place where each pupil was working on the problem next in line for him. Many of the abler pupils were called upon to help the others constantly, and were probably aided considerably in their own understanding of the subject. The pupils many times stayed after the class or came back at vacant periods or after school in order to get special aid from the teacher.

Class Discussion

Attempts were made several times during the period and during subsequent periods to discuss certain phases of the unit in the class as a whole. This was usually rather unsatisfactory, because of the different rates of progress of the pupils. This discussion, to claim the interest of all or of most of the pupils, will probably have to be a summing up process of material they have all passed over. Discussion in groups of four or five girls could probably be worked out with profit. This is one phase of the project that could probably be adjusted much more satisfactorily.

Testing and Marking the Pupils

The marks for the pupils daily work are indicated on the work sheets. The standards set by these cumulative marks, at bottom of the work sheets, were not felt to be entirely satisfactory. For example, on one of the work sheets used, practically all the pupils attained an *A* rating. This was partly corrected in the later work sheets by assigning more and harder problems for the higher marks, but it could be more properly handled by having the problems cor-

rectly graded as to difficulty. It might also be of interest to know that the pupils generally refused to take less than a *B* rating on any problem, they preferred to come back and try again after more study. At the end of the period a test was given to all; this was partly a new type test but not very well worked out. However, it gave a fairly normal distribution and agreed very well with the teachers estimates. The final mark for the period was figured from this and from the marks on the work sheets, used as daily marks.

Results

The results of the experiment were felt to be generally satisfactory. The pupils probably worked many more problems and understood the subject matter of geometry much better than they would have by being taught as in the first half of the year. This opinion is borne out by the fact that the marks for the period were in general higher than in the early part of the year, although as already stated they were fairly normal in distribution.

Some of the pupils who obtained credit on most of the problems on the work sheet fell very low on the test, some to *D* or *F*. In most cases these were the pupils who had been making poor marks all along. This is to some extent inherent in the checking system used. These people had evidently gotten credit on the problems without understanding all of them thoroughly.

The class reaction was favorable on the whole. This opinion was formed from the general attitude of the class and from expressions by individual members at various times. Unfortunately no systematic way of finding how the pupils felt about it was used.

Mistakes

The following things may be considered mistakes that were made and should be of aid in using the plan in other situations. The reader will recognize that some of them are mistakes inherent in the method of individual instruction and will probably be hard to correct or even to minimize. However they should be kept in mind when trying to use the method. Others can be remedied rather easily.

1. No time was given for the pupils to sum up and see a group of problems, corollaries and exercises as a whole. The pupils worked too much from day to day and were too often satisfied with a mere mark on their work sheets.

2. No tests were given until the end of the period and as a result many of the pupils did not realize where they stood. They did not even realize that their work sheet records were unsatisfactory.
3. Some were allowed to explain too many problems at the last of the period when they realized that the time was short and they had not attained a very good standard. As a result they did not understand them as well as they should have otherwise.
4. The monitors had their own work and their monitorial work to do without receiving extra credit for it.
5. Pupils were allowed to pass problems without understanding them. This was of course the fault of the inexperienced monitors in many cases but some could also be laid at the feet of the teacher.

Recommendations

The following recommendations are submitted as ways of correcting some of the mistakes listed above. Some of them were quite effectively applied in subsequent periods.

1. Two half periods each week to check up and get an idea of the relations between the problems.
2. A definite testing program should be worked out. This could be done with little loss of time on the part of the pupil if the teacher used the new type tests and had them made or selected before hand and definitely scheduled.
3. Post marks at stated intervals, say at the end of each week. Let this list contain both the class mark and the average of the class and test work.
4. Have each pupil explain at least a certain percent of his problems to the teacher, perhaps one in five, in order to prevent him "putting it over" on one or more monitor by some method.
5. Get some way to equalize the load on the pupil monitor. It is true that they learn the subject matter better by being monitors and they must understand all the problems thoroughly in order to check others, but they should at least get extra credit for the work they do in this capacity.
6. Limit the number of problems the pupil can explain for credit during the last few days. This would prevent some lower class pupils "railroading" themselves into a higher class in the last two or three days of the period.
7. Secure a text written for individual instruction. Many of the

pupils' difficulties were caused by not being able to understand a book written for the group instruction method.

8. Something should be done to determine what standards should be set. We do not know just how many problems a good student should be able to work in a week's time. Another phase of this problem is the relative difficulties of the given exercises.
9. See that the pupils actually write out a certain part of the problems in "full form." This is based on the assumption that a pupil gets a much clearer and more lasting impression of a problem in geometry if it is written out in full.

Notice to National Council Members

It is evident that in preparing a register of members of the National Council of Teachers of Mathematics some names are sure to be inadvertently omitted. There are several reasons which might be given why this is so. However, the mere stating of reasons will not help. We have already discovered some omissions and also some incorrect addresses. Will you please look at your copy of the Register and if you were a member when the Register went to press, and if your name has been omitted, or if either your name or address is incorrect, please let us know at once and we shall try to publish in an early issue of THE MATHEMATICS TEACHER a list of corrections. Such corrections will enable those whose names have been omitted or incorrectly stated to receive the benefits that may derive in having such a correct directory of our complete membership.

—THE EDITOR

Socialized Mathematics

By W. J. KLOPP, *Director of Teaching, Woodrow Wilson High School, Long Beach, California*

MUCH IS BEING WRITTEN about mathematical objectives to be realized in courses offered in secondary schools and much stress is laid upon the modern trends in mathematical teaching outcomes.

An analysis of teaching procedures in mathematics reveals a great need for better technic in teaching mathematics in order that pupils may gain a greater per cent of mastery of the fundamental tools for use in mathematical situations and ultimately for a larger appreciation of mathematical knowledge as it applies to current, social, and economic problems.

It would indeed be unpedagogical to presuppose that objectives of any kind can be realized to the full without a basic knowledge of the fundamental elements entering into such realization.

One outstanding fact revealed by a recent survey was that pupils fail to retain fundamentals of mathematics long enough to be effective in their progress. Take the fundamental operations as applied to algebra for example. Unless the principle as well as the process of addition, subtraction, multiplication, and division are mastered by all pupils it is obvious that there cannot be accelerated progress in the application of these principles in the more complex assignments, hence there must be a cumulation of weakness experienced through the course resulting in ultimate failure to succeed before the end of the course. Such failure is usually ascribed to low I.Q. or lack of interest in mathematics. This is not necessarily a fair conclusion since in an experiment conducted in the Woodrow Wilson High School of Long Beach extending over a period of three semesters it has been shown that by the use of review technics, success in the mastery of fundamentals can be attained by pupils varying in I.Q. from 75 to 165. The correlation between I.Q. and progress under the new technic was only .41 and between I.Q. and the results of a standardized test was .42, while the correlation between progress by the new technic and the results of the standard test was .82. In all three cases the probable error was less than $\pm$ .03.

It is further revealed by the experiment that under the traditional

method of teaching algebra pupils lack the tools for success in advanced mathematical courses. Sixty-six junior college students enrolled in mathematics were tested on the fundamental operations and 50 per cent solved less than half of the problems correctly, none receiving a perfect score. In testing 125 junior college students on the language of algebra covered the first three weeks of first year algebra, the range in scores was from 4 to 34, and the 125 students scored but 66 per cent on the test. Later 75 junior college students were given the Columbia Research Test in first year albegra and the median score was zero.

For further substantiation of the claim one may cite the results of Everett's study in which 46 problems on algebra skills were submitted to 1,200 high school pupils having completed one year of algebra. Fifteen per cent of the 1,200 pupils solved the first problem correctly leaving 85 per cent who failed, 21 per cent solved the second problem; 22 per cent solved the third, and 23 per cent solved the fourth; less than half of the 1,200 solved any one of the 46 problems involving the fundamental skills essential to success and progress in advanced mathematical situations. It is needless to say that such a picture cannot inspire one to feel that the socialization of mathematics can solve the problems of mathematics and assure the realization of any objectives.

Another study, involving 256 cases, in which the pupils were tested by the Hotz Scale at the end of seven months, showed that 25 per cent worked less than 2.9 of the problems correctly which was less than the norm for three months of algebra, and the norm is considered too low by the average teacher of algebra. With these facts before us we must focus our attention upon teaching technic rather than a new or different curriculum. We must think more seriously about the outcomes of method and less about the outcomes of content.

The experimental technic emphasizes the following remedial procedure:

1. An agreement upon the fundamental elements to be mastered.
2. More effective diagnosis of the pupil's knowledge of these fundamental elements.
3. The establishment of self-administered drills and tests.
4. Provision for rate of pupil-performance either through or-

ganized material or by classification on the basis of the results obtained through diagnosis.

An experiment of ours first concerned itself with third semester algebra review for which eight units were worked out on the remedial plan. These units involved the langauge of algebra, the fundamental tools of algebra and their application, and the solution of problems and graphs. It was soon obvious that pupils performed at varying rates. It was also observed that fewer pupils were dropping out of the course. In fact only two out of 102 failed to continue the course and only two failed to complete the minimum essentials. We therefore had less failures, no repeaters, and all who completed the assigned units did so on the mastery basis since none were advanced from one unit to another without passing the mastery test 100 per cent.

A study was made of the errors made by pupils in taking the mastery tests and it was found that only 12 per cent of the errors were errors of accident, 88 per cent were errors of principle.

The experiment has been extended this year to include all classes in algebra and the third semester class has made such signal progress that ten per cent of the class completed the semester's work in eight weeks and were transferred into a junior college mathematics class in which they have been placed with the average of the class in ten days.

The introduction of these technics into beginning algebra is yet too young to make any prophesies but a cursory test of their effective value has been made by the use of the Hotz Scale at the end of ten weeks in which a class of 50 scored an average of 18 with the lowest score of 14 and the results were compared with a second semester class of 200 which made an average score of 16 and their lowest score was four. These results indicate that the weaker pupils at least reached their level and the superior pupils increased their rates of performance. And this is as it should be. It is hardly sound philosophy to assume that pupils of high school age and in high school are not able to master the fundamentals of mathematics and apply them in common mathematical situations. If this were not true then how can we hope to aid the weaker through a program of socialized mathematics when they do not have the tools to apply to social-mathematical situations.

Mathematics as a Fine Art

By James Byrnie Shaw
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Mathematics studies a vast world of objects and their transformations, their static and dynamic characters, intricate and tangled, yet systematic and ordered to a degree far greater than that of any other known world. This world has been created by the mind of man, partly in order to enable him to handle the world of phenomena, but mostly for the elegant beauty of its objects. The mathematician is an artist whose material is more subtle than stone or paint, tone or words. Of late he has made a thorough study of what he really does and what he really studies, and has concluded that he has always exercised a creative function of his mind. He has come to see that the rainbow which lures him on for the pot of gold is one of his own making, and the gold he finds was fused in his own crucibles. Indeed he is able to point out to the scientist that most of what he too is concerned with, is of the same character, and that he too works for the aesthetic satisfaction he gets. It is this character of mathematics which we shall consider for a few minutes. And first let us look at a few peaks in its history.

There was once a brilliant genius who watched the stately march of the stars from the mountain tops of Samos, or drew diagrams of geometry on the sands of the Icarian beach for the blue waves of the sea to erase. He made magic squares of numbers, or at times charmed his circle of students with the music of his lyre. To him mathematics was music and music was mathematics. In the cube he found symmetry of form; in its 6 faces, 8 vertices and 12 edges, he found the numbers for the fundamental tone, the quarter, and the octave: harmony of form and harmony of sound combined. He had the intuition that the world was built to music and therefore to mathematics. Triangles, squares, pentagons, cubes, icosahedra, were certainly incarnations of number, so were the tones of the lyre, why not all nature? The heavens must rotate in perfect circles, and to him that had ears to hear, the music of the spheres was evident. Had he sensed the relation of the modern group to the forms of the flashing crystals dug from the mountain caverns, he would have had all the more reason to

believe that the world is number incarnated. He saw mathematics building a tremendous symphony out of the orchestral music of nature. Across the sweep of twenty-five centuries we see Pythagoras' majestic figure, and appreciate his far-seeing vision.

But the use of merely integral numbers turned out to be insufficient for the demands of the intellect, and after it was found that the diagonal of a square could not be a whole number no matter what whole number was taken for the side, the idea of the irrational number was born, and the range of arithmetic was extended beyond the farthest horizon. Then two milleniums flowed past and there was no further evolution. But the day came while the crusaders were battering the walls of Jerusalem. On the banks of a lotus pond in India sat Bhaskara watching the swaying lily-stalks, and the golden bees that settled in the flowers, while he wrote his treatise called "Lilavati"—the science of the Beautiful. In trying to solve quadratic equations he found it necessary to view the world of number in a mirror, and to find for every number its left-handed twin, the negative. This was startling and, as with all new ideas, repugnant to the ordinary conservative mind. However in the course of time, even the world of commerce found that the new numbers could be used in credit accounts and they thus became respectable. Some centuries passed and the world woke from its lethargy and began to hunt adventure. While DeSoto was exploring the new world of geography, another adventurer wrote the Ars Magna, and the square root of minus one, the most daring conception of the human mind that had yet burst into bloom, put up its solitary stalk, and spread its petals to the renaissance of thought. It was shunned for a long time as the work of a sorcerer, but it had come to stay, and since Steinmetz showed that it could be used by traction lines to increase their profits, it too is now respectable, and not the product of magic. It is beyond all doubt a creation of the mind emanating from no previous data. All numbers became at once not simply twins, but rayed like stars, for to each value corresponds an infinity of rays. Three hundred years later, the brilliant Hamilton announced a new inflorescence of number, which first appeared clustered in tetrads of imaginaries, then bestarred with imaginaries of bewildering types, hypernumbers in infinite variety. The mathematician knows now that he can at will, like Prospero, summon these magic flowers by a mere wave of his wand. But unlike Prospero's cloud-capped towers they do not vanish

with a gesture: rather they spread luxuriantly and bloom superbly, and from them we may gather elixirs and perfumes.

We turn now to geometry, back to the age of the Seven Wise Men, where started the glory of Hellenic thought. Whencever the seeds may have been imported, in the atmosphere of Greece there flourished the beauties of thought gathered together later by Euclid. It became as rank heresy to question the truth of geometry as that of theology. A knowledge of geometry was a prerequisite to entry to Plato's Akadême, and he is even credited with the idea that the evolving universe is the geometrising of God. The foundations of everything in existence were held together by the irresistible logic of geometrical constructions. A few timid souls tried to justify one of the postulates of Euclidean geometry, but it was reserved for Lobachevsky, bolder even than his contemporary, the great Gauss, to create a new world by creating a new geometry. This is the world of many parallels, a world in which areas depend upon the sum of angles, similarity vanishes and every figure is unique. It is a world of individuals, a true democracy, not a standardized society. A little later Riemann added a whole host of new worlds by creating new geometries. In some of these the inhabitants are far more than individuals, they are eccentric individuals. Modern relativity is an outcome from that, and once again a fantastic creation of the mathematician's creative imagination has seized the world of physics, and made it over. We need not dwell on four-dimensional spaces, N-dimensional spaces, and even spaces with infinitely many dimensions. These are exotic orchids of the tropics of the human mind, brilliant with unpaintable hues, perfumed with fragrances no chemist can extract, beautiful with forms no draughtsman can delineate. Their chief reason for existence is the fascinating beauty they possess.

We can only glance at the history of analysis. The idea of function is prominent here, and we realise the immense significance of the fundamental notion of Heracleitos, the philosopher of change, the high priest of the mathematical transformation. For instance the simple function x^2 stretches the chain of beads on the x axis which we call the integers into a new system that gets farther and farther apart as we proceed along that axis. We find periodic rythm changed into cadences, for the intervals get longer and longer, but under a definite law. We find the function of a complex variable transforming the plane covered with a lattice-work of lines, into a new plane covered

with a lace-work of curves. Again, with forceps whose delicacy is infinitely greater than that of the tools of the most skilled jeweler, the mathematician strings numbers like glittering beads, on golden wires whose curves are inexpressible in material form, and which are invisible to the eye. These curves have nowhere any direction; they may go through every point of a square area; they may leave empty an infinitely numerous set of positions; they may leave every bead at a measurable distance from its neighbor, and yet occupy every position; they may have direction but nowhere a definite curvature—in short they are of such intricate workmanship that it is impossible to deduce any of their properties by inspection.

We might look over the history of mathematics in every age and follow its evolution through the revolving cycles, and we should always find the same conclusion. We would always come to the vague, the chaotic, the tangle, the unformed, but under the magic wand of mathematics we would find the vague assuming form, the chaotic appearing in order, the tangle turning into a beautiful lace, the unformed showing first a line, then a net, then a structure. We find this taking place because the mathematician creates lines of connecting new gossamer webs that run from point to point, then unite into definite structures. Indeed the modern scientist has discovered to his dismay that what he is studying is really not the world of phenomena at all, but the webs spun by the mind to hold these phenomena together. And the mathematician does not stop with phenomena, he creates structures for the pure joy of creating, whether any phenomena can be attached thereto or not. His guiding principles are those of all artists. He is carrying out Sandburg's definition of poetry as the "achievement of the synthesis of hyacinths and biscuits." For a long time he has been mostly interested in the biscuits. He had a few years ago almost sold his birthright for a mess of pottage, but in the last few years he has turned toward the other side of his work and is more and more cultivating the hyacinths.

The difference between poetry and mathematics is mainly one of subject matter. Whitehead said recently that mathematics is the greatest invention of the human mind unless it is perhaps music. A writer on Darwinism said that it is impossible to trace any connection between either the mathematical faculty of man or the musical faculty, and the struggle for existence. The greater part of mathematics is fiction, more fiction than any maker of novels has thought of, and

more harmony than all symphonies. It is the product of the ceaselessly creative activity of the spirit. And mathematics bakes biscuits. Said Poincaré, "Mathematics has a triple end. It should furnish an instrument for the study of nature. Furthermore it has a philosophic end. And I venture to say it, an aesthetic end. It incites the philosopher to search into the notions of number, space, and time; and above all, adepts find in mathematics delights analogous to those that painting and music give. They admire the delicate harmony of number and form; they are amazed at the unexpected vistas opened by a new discovery; and has not the joy they thus experience an aesthetic character even though the senses have no part therein? Only the privileged few have a chance to enjoy it fully, it is true, but is it not so with all the noblest arts? Hence I do not hesitate to say that mathematics deserves to be cultivated for its own sake, and theories not admitting any application to physics deserve to be studied as much as do the others". The practical advantages of mathematics have been urged so long as its reason for existence, that it seems time to turn back and once more look for the hyacinths and not the biscuits. The characters that charmed Pythagoras and Plato, Descartes, and Spinoza, and all the long roll of those who have studied Mathematics as a pleasure, from earliest times up to the present are certainly those which would furnish reasons for mathematics. We need more mathematicians today like Pythagoras, Omar, Descartes, Hamilton, Sylvester, and Poincaré.

II.

Come with me for an inspection of the principles of Art and how they appear in mathematics. We may put them under the heads: Design; Symmetry; Rythm; Harmony; Unity; Ideality. These appear in all the arts, whether in the great cathedral, the exquisite statue, the awesome painting, the haunting symphony, the aching poem, the thrilling drama, or the bewitching dance. We can only touch lightly here and there for instances.

1. Design

Design means order and composition. Order relates to the way things are put together and composition to the general pattern. In a piece of music we find order in the melody and the working out of the theme, composition in the succession and the harmonising. Then we

have such comprehensive designs as symphonies. So too in a poem, we have an order in the succession of ideas and what they suggest, and composition in the development of the theme. The same is true of all the arts and in particular of mathematics. A few examples will make this clear.

1. Every rational fraction is the quotient of two integers, and no irrational can have a design of this type. Every number not a prime itself has but one set of prime factors whatever their order. This principle of unique factorisation is so highly valued that it is extended further into the realm of algebraic integers, even if it is necessary to introduce ideal numbers. For instance we may factor 6 into 2 times 3 or equally into $1+j$ times $1-j$ where $j^2+5=0$. Neither of these four factors can be broken up into algebraic factors which are either positive or negative integers or numbers of the form $x+yj$. If however we represent the square root of 2 by p, and $\frac{1+\sqrt{-5}}{\sqrt{2}}$ by q, and $\frac{1-\sqrt{-5}}{\sqrt{2}}$ by r, the last two not algebraic integers, then we have $2=p^2$ $3=qr$, and $1+j=pq$, $1-j=pr$. This shows that the correct design for 6 is ppqr. The two different ways of factoring are merely different groupings in pairs of the ideal factors. Numbers may be expressed as sums of squares, sums of cubes, sums of fourth powers, and sometimes in more than one way. But there is a definite plan in each expression. We may also have certain designs that cannot be used, as for instance the expression of the sum of two cubes as a cube. In the sets of three integral numbers which furnish the sides of a right triangle, why should not the student learn that the largest is always the sum of two squares, the other two are the difference of the same squares and twice the product of the two numbers. Also that one of the perpendicular sides must always be a multiple of three. Why should the fifth powers of numbers always end in the same figure as the numbers? Why should fourth powers end only in 1, 5 or 6? The tests for divisibility are theorems in design. Every long multiplication of numbers is a problem in composition, for the various partial products go together in a very definite way.

2. When we look at elementary geometry we find design expressed by composition and order everywhere. The diagonals of a parallelogram always bisect each other. The perpendicular at the middle point

of the base of an isosceles triangle goes through the vertex and bisects the vertical angle. Any quadrilateral is so designed that the midpoints of the four sides are vertices of a parallelogram. A regular polygon is a composition of isosceles triangles and all the theorems flow from this composition. A secant through a point cuts a circle so that the segments always have the same product, which is a characteristic number for the point with reference to that circle. The circle is given by this definite design. A system of circles through two points have order but still make a composition, and this includes the conjugate system which surround the two points and cut all the first system at right angles. The radical axes of three circles meet in a point, and that is the center for the circle that cuts them all at right angles. A system of circles all orthogonal to one circle must have all their radical axes going through the center of the given circle. A system of parallels is a composition, and whatever is the set of ratios of the segments in which they cut another line, this is the same set of ratios for every line. The six tangents common in pairs to three circles, give six points of intersection and these lie on only four lines, the axes of similitude of the circles. This composition and the common center of the radical axes, furnishes the means of drawing the eight circles tangent to the three. This composition was practically the end of the classic mathematics of the circle. Modern geometry has many elaborate designs, and they all exemplify the artistic quality of the subject. Why should not the student have a chance to enjoy this beauty?

3. We find design everywhere in elementary algebra. The factorisation of expressions; the structure of the coefficients in products, as in binomial and polynomial expansions; the addition and the decomposition of fractions; the sums of finite series; the roots of equations as rational designs of a single irrationality; determinants; systems of dependent equations; symmetric expressions; Sturm's functions; all the long list of indentities, give abundant instances of design. Why should not the student have a chance to see this fascinating phase of the subject?

4. If we pass to higher subjects, the same is true. In the characteristic curves of differential equations we find a composition which is the essence indeed for what the differential equation signifies. In the invariants of higher algebra, in groups, in congruences of forms, in the theorems of expansions, everywhere we have design. It is of an intellectual type it is true, not as brilliant as the color scheme of a

modernistic painting, nor as strange as some of the compositions of modern musicians, but no less a delight to the aesthetic sense. A student who passes through this realm without seeing these elements of beauty has been defrauded.

2. *Symmetry*

Symmetry includes balance and value. Balance is that feature of a design which places its constituent parts so that they furnish an equality in the different directions. Two equally prominent parts are placed equidistant from the central part; if one is more prominent than the other it is placed nearer. In a picture we may have, for instance, an Indian chief in the foreground adoring the setting sun, and to balance him, purple cliffs in the vague distance. Value is that feature which makes one portion more or less vivid or faint. Purple mountains fade to amethyst as distance increases.

In a reflection in a silvered globe we find objects of the same size actually appearing smaller but farther back in the mirror. In a landscape colors that are bright for near objects are pale for distant ones of the same actual color. A triangle with its inscribed and escribed circles is an example of balance. The largest outer circle has its center farther away and its contacts farther off. Symmetry is shown in the division of a triangle by its medians, by its bisectors, by its altitudes, by many other lines which pass through one point, the center of the symmetry. Much of the modern geometry of the triangle is a series of studies in balance and value. The theory is elegant, was even called an elegant trifle by Professor Tait, too much interested in biscuits. We may say with some truth that the distance of the center of a circle balances the approach it makes towards straightness, and if it is straight the center has to be at an infinite distance. A perfect symmetry with equal values is shown in the reflection in a straight line, A series of such reflections, as in kaleidoscopic designs, is a multiple symmetry. Inversion in a circle gives symmetry and change of value. Relationships of pole and polar are symmetries, and in case figures are reciprocated, we have balance and value playing an important rôle.

2. In algebra we find balance in the terms of a series which converges, the farther terms being of smaller magnitude. We find symmetry in the relations of roots and poles corresponding. We find balance in the signs as given by Descartes' rule, and in Sturm's functions. In the roots of the Hessian of a cubic we find two points on

which we may balance the structure, and upon which depend the invariants. The cubicovariant is a symmetric quantic (in the artistic sense of symmetry of course) for the cubic itself. The parallelism between the symmetric expressions of the roots of any equation and the same expressions with the proper differences of the roots which we use for constructing the Sturm functions, is a beautiful example of symmetry.

3. Analytic geometry is itself a brilliant example of symmetry, and when Descartes saw that every theorem in geometry was also one in algebra and vice versa, he opened a new world in mathematics. An equation of a curve is symmetric to the figure of the curve. And the envelope equation is symmetric to the locus equation. Every form used in the algebraic part of the analytic geometry is an algebraic statement of an invariant for the figure under those changes which Euclidean transformations permit, that is to say, under rotations and translations. The vanishing of any such expression is an equation which picks out a particular curve or distance, or angle.

4. We may consider that every case of isomorphism of any sort in mathematics is a symmetry and in it we may find the balance and the value character of what enters the isomorphism.

When we match algebraic or analytic expressions against the ideas of physics, we have a case of symmetry, the larger subject now being applied mathematics. Indeed the whole of modern physics seems to have become the mathematical expression entirely, as far as actual development is concerned. Physics is translated into mathematics, then conclusions are drawn which are translated back, and if the parallel laboratory procedure is missing it is no matter of importance. The whole of calculus is a balancing of what takes place with what the limit implies. Even the apparently frivolous task of finding out in any situation how many of the conditions may be removed and leave a stable structure, is the appeal of the light and airy as balanced against the solid and well-braced. Even logical proofs may exhibit features of balance and value, as shown in theorem and corollary, in major and minor parts of the proof.

3. Rythm

Rythm appears in periodicity and cadence. It corresponds to the rythm in life, to the repetition of experiences, to the ebb and flow in the thrill of youth. It always suggests change, movement, but with a memory of what has gone before. The waves that sweep the beach

the beat of sea-gull's wings, the rising and falling breeze, the changing cycle of the seasons, the rippling lines flowing down the mountain side, the long plume-streamers of the clouds, the flicker of the sunlight in the pool where the trout lurk, the rythm of the rainbow hues, birth and death, growth and decay,—everywhere the world is built to rhyme. Whitehead makes it the life principle of his organic realism.

1. We find rythm early in arithmetic. The rational number expressed in decimal form is an instance, and on any other base than ten we would find the same thing true. If the denominator contains no other factors than 2 or 5, the decimal somewhere has a repeating chain of zeros without end. If there are other factors, the repetend contains other figures. Rationality and rythm go hand in hand.

2. In the Galois theory we have a beautiful example of rythm. The resolvent equation is defined as one such that if we start with any root, we may express the others rationally in terms of this one, the coefficients, and the arbitrary numbers used in constructing the system of symmetric expressions of the resolvent equation. And whatever root we start with, the same forms will give us all the others. The forms themselves constitute a group, and the whole theory of groups is a study of higher rythm patterns. We can find cadenced patterns if we include the infinite groups. Recurring finite series also give examples of cadence.

3. In elementary geometry rythm is frequent. The various designs in architectural work, the forms of crystals, the various star-patterns, the relations of similar figures, inscribed figures with figures inscribed in them, poristic diagrams, harmonic ratios,—the list is long.

4. Automorphic functions give an increasingly subtle rythm as we pass from the simply periodic function, through the elliptic functions, the modular functions, into the most elaborate cases. In the fundamental polygon used with automorphic functions and its subdivisions, usually infinite in number, we see readily why rythm appears here. In the Sturm-Liouville functions we find cadenced rythm in passing from one root to the next, for there is repetition of general features with increase of detail.

5. Mathematical induction is the crystallization of rythm in the form of an intuitive method. It is most important, for in one way or another it appears everywhere in mathematics, and if we consider that it is a statement of the rythm that occurs under various transformations of the initial concept, we might make it the most important

method of all mathematics. For instance if we prove a theorem for a given isosceles triangle, we can see the same theorem repeated as we change the altitude of the triangle, and then as we stretch out the base from its middle point, so that we come to see that all isosceles triangles are rythmic examples, infinite in number, of the same theorem. This brings us to a rythm based upon infinitesimal changes, and we must of course introduce it into the theory of continuous groups as well as into discontinuous groups.

4. Harmony

Harmony means the consistency of the whole. Of course we sometimes admit complementary contrasts to enhance the effects, but after all they are part of the consistency, a feature clearly seen in the singular solutions and singular points in mathematics. We would not introduce a Charleston jazz dance in the middle of an oratorio, nor in a desert painting would we show a ten story skyscraper. To put a battle into a villanelle would ruin the harmony. But we may have a summer shower in the pastoral poem, and a desolate mesa would harmonise with the desert. Harmony does not mean weary sameness, for it permits variety, but there must be a fitness of what is selected to compose the work. It is trivial to say that mathematics demands consistency, for it would be impossible without it. Indeed mathematics is the superb example of consistent human thought. But we may not always find harmony of the highest type. For instance to study an Archimedes spiral by rectangular coordinates violates the principle of harmony. To use metric methods on problems essentially projective violates harmony. To introduce calculus into purely algebraic theory is a flagrant violation. For instance, to call the sum of the products of the factors of a polynomial taken n—1 at a time, the derivative of the polynomial, merely because it happens to be the same form, breaks harmony. It introduces the idea of limit which does not fit the algebraic notions at all. As an instance of contrasts in harmony, the singular solutions of differential equations fit the harmony of the equation, but are in contrast to the characteristic curves which belong to the equation. Yet they envelope these curves and tie the whole together into a unit. An entire development of a theory is a case of elaborate harmony, as in Euclidean Geometry, our Lobachevskian Geometry.

It is internal harmony which makes it possible to apply mathematics

to nature. For if this harmony did not exist sooner or later we would come to an inconsistency. We have on the other hand the contrast of the precise exactness of mathematics and the foggy inexactness which must forever reign in natural science. But in the end as Poincaré pointed out science too is possible only because it is a search for harmony in the outward world.

5. *Unity*

Unity implies coherence and closure. Each work of art is in a certain way complete in itself and has one central theme holding it together. When a story has within it another story, or a play has a play within it, or when a painting can be cut into two paintings, or a musical composition consists of two different pieces, they lack unity. When a play simply stops, or when a story leaves one wondering what happened next, they lack closure. We find these elements in mathematics. The quadratic in two variables furnishes theorems for all conic sections, and these curves are found to possess a unity that might not be suspected otherwise. When we see that the solution of simultaneous equations is always accomplished by determinants, no matter what the details of method may be, we have found the unity in them. When we see that any method of solving a polynomial in x consists essentially in finding the factors we have the unity underlying. Galois brought in unity when he said all the roots of an equation were expressible in terms of a single irrationality. In the presentation of subjects unity may be violated and the student left with an incoherent set of notions. In elementary geometry for instance a collection of isolated proofs is of little importance. But when the various theorems are all seen to flow from one central idea, unity is preserved. A great many theorems can be made to depend on similar triangles, and these upon the fact that their angles are equal. Once this is seen, the unity of the proofs gives a momentum to the learning. Other theorems reduce to properties of parallelism and this simplifies the proofs and makes them natural. A fine example of unity in method is in Reye's Geometrie der Lage, where the projective synthetic methods, using pencils and their intersections, furnish a very coherent treatment. The great need today in the theory of integers is a unifying principle. The use of linear operators as the basis for all work in expansions arising from integral equations, differential equations, and linear functional equations in general, brings out the essence of the subject,

the internal structure as no other method can. Also every development should be properly closed so that it is complete in itself. Not that everything can be said about the most simple problem but that what is said should be pushed to a natural boundary. To leave quadratics with nothing more than a formula for solution is to leave the subject unclosed.

6. *Ideality*

Finally we come back to the standpoint from which we started, in the most central principle of art, that of ideality. Art does not copy nature or humanity, it always has something new to suggest, some new vision for the future. A photograph however accurate may not necessarily be merely a photograph, but it is not artistic if it is merely a photograph. Even a biscuit may be a work of art, reflecting in every quality the character of its maker. The ideal brings out something that we have not seen before, and opens vistas and new paths to traverse. Realism that is artistic is not merely a faithful reproduction of nature or life, for it must suggest by means of the phases selected something beyond nature or something profound in life. Of course every great artist sighs over the way his creations fall short of what his vision contained, but he presents what he has power to present, so that others may catch the invisible spirit. It is this element of everlasting youth, of eternal spring, which makes art worth while. There is ever a pathway down which the west wind blows, something beyond the seen, and ideality is a spirit which always beckons us to other fields.

In mathematics there is always a flowering of new plants, some bringing forth blossoms of exquisite beauty. It has within itself the fountain of youth, for it never grows old. The ideal worlds it creates are far more interesting than those to be found in terrestrial affairs. Behold the strange and delightful modular geometries of Dickson, the bewildering space that Einstein has introduced us to, the pulsating life of the world of modern phsyics! Wander through the mazes of nilpotent algebras, the tangle of discontinuous functions, the intricacies of non-archimedean arithmetic! Mathematics shows its essentially artistic character in its sublime play with these fanciful creations. For every artist plays; he creates for the joy of creating. The student who has had no glimpse of this but looks upon mathematics as the kitchen maid who gives him something to eat, instead of a delightful playmate full of enchanting dreams, has been

cheated in his mathematics course. The student who has had no chance to look over the great gallery of mathematical pictures but has been compelled to learn merely the mechanics of their painting, also has been worse defrauded.

The original meaning of "mathesis" was "knowledge," and "poiesis" was "creation," and the twain are one, mathematics and poetry are essentially the same.* There is no real knowledge in mere acquisition, there must be a creative reaction of the spirit; and there is no creation without a new insight into the universe of nature and spirit. This is the real knowledge. Says Bertrand Russell, "Remote from human passion, remote from the pitiful facts of nature, the generations have created an ordered cosmos, where pure thought can dwell as in its natural home, and where one at least of our nobler impulses can escape from the dreary exile of the natural world." Pringsheim once said "One would be astonished to find that mathematics, which to the common mind is arid, is united most intimately with the blue flower of Romanticism." Sylvester said "Music is the mathematic of sense, and mathematic is the music of reason." Nature herself is ideal, for whence come the suggestions of unseen vistas which the scientist has, save from Nature herself? And we all know that if mathematics keeps close to the problems of Nature, there will always arise new ideal objects for thought, and these usually transcend the mere hint that Nature gave us. We find that Ideality, like all the other principles of Art, is intimately woven into the texture of the universe. Mathematics is the synthesis of hyacinths and biscuits, so that the biscuits should be artistic biscuits, and the hyacinths should not scorn the banquet table. Ideality makes mathematics sister to all the other arts, whether they build cathedrals, model Psyche, paint the desert, sing the verses of Sappho, weave three sounds into a star, or dance the bacchanal of eternal youth. In the mutual interchange of ideals we find the wine of inspiration, the new vision of grandeur.

"For truly as thou sayest, a Fairy King
And Fairy Queens have built the city, son;
They came from out a sacred mountain-cleft,
Toward the sunrise, each with harp in hand,
And built it to the music of their harps.
And as thou sayest it is enchanted, son,

* See the new book of Scott Buchanan on "Poetry and Mathematics.

For there is nothing in it as it seems,
Saving the King; though some there be that hold
The King a shadow and the city real;
Yet take thou heed of him, for so thou pass
Beneath this archway, then wilt thou become
A thrall to his enchantments, for the King
Will bind thee by such vows as is a shame
A man should not be bound by, yet the which
No man can keep; but so thou dread to swear,
Pass not beneath the gateway, but abide
Without among the cattle of the field,
For, an ye heard a music, like enow
They are building still, seeing the city is built
To music, therefore never built at all,
And therefore built forever.

III.

In conclusion then I bring you a challenge, a challenge to a new adventure in the presentation of mathematics. I challenge you to undertake to make equally prominent the applications of mathematics, and the beauty of mathematics. I challenge you to make as many exercises which will show the beauty of mathematics, as you make to show its use. Set forth exercises which require design, which exhibit harmony, rythm, unity, ideality. Teach the student that mathematics walks upon the earth, but her face is to the stars. Teach him some proofs but do not keep him from seeing the glorious things created by mathematics simply because he cannot for lack of time or for lack of background follow all the proofs. Present the subject intuitively, and synthetically, and let him acquire a knowledge of it by the route of the artist rather than that of the logician.

Is it a vain hope that under such teaching a generation of students would find an inspiration now lacking? Is it too much to hope that there would spread abroad a different notion of mathematics, so that its real wealth of charm would be recognized, and instead of seeming to be a wilderness of nightmares and terrors it would be a fairyland of flowers and murmuring brooks? I bring you a challenge to undertake what seldom has been undertaken to make mathematics *fascinating* and *easy* to understand, a *joy* to study, a *satisfaction* when learned, *play* and not work. I challenge you to

connect mathematics with every phase of life, to open the door through mathematics to the solution of the riddle of the Sphinx.

"I am the poet's dream,
The harmony
Of all music and of all the spheres,
The painter's visionary gleam,
The illumined page,
The storied love and heritage
Of all the years."

National Council Yearbooks

The National Council of Teachers of Mathematics would be able to dispose of past and future editions of the Yearbooks if the members of the organization all over the country would only take a little more personal interest in helping us with the distribution and sale of them. If you can possibly help by selling some of the Yearbooks to your own group send in your order at once to the Bureau of Publications, Teachers College, Columbia University, 525 West 120th Street, New York City. You may return any copies you do not sell. A reference to the contents of the Fifth Yearbook will be found in one of the advertisements toward the back of this issue.—THE EDITOR

Consumer's Uses of Arithmetic

By C. N. Shuster
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Dr. A. O. Bowden's dissertation, "Consumer's Uses of Arithmetic," has received so much publicity that a review and evaluation in The Mathematics Teacher seems not only fitting and proper but quite necessary.

Dr. Bowden says, "The writer collected copies of all arithmetics now on the market and those that have been in use within the last quarter of a century." From these texts Dr. Bowden selected 345 problems that he considered "typical." A second list of 123 problems was taken from the first list and sent to 682 parents who were asked to answer this question, "Aside from business or vocational needs have you ever had occasion to use problems like these since you left school?" As a result of this voting, Dr. Bowden attempts to find an objective "in arithmetic for the present and coming generation," by discarding all problems that received less than 25% of the votes.

It would be an extremely difficult task to collect copies of "all the arithmetics now on the market and those that have been in use within the last quarter of a century," for this would include practically all the arithmetics written or printed in the last fifty or more years. During this time there have been so many changes in arithmetic that any list of problems selected from such a heterogeneous array as all these scores of texts would form would be quite worthless for any study. Such a list would help us to evaluate neither the arithmetic content of twenty-five years ago nor that of today. In fact, the content in arithmetic is changing so rapidly that practically all the publishers have been forced to revise their arithmetic texts within the last five years. To have any value as a study that aims to evaluate modern content, the problems should be selected from those recent texts that have a sufficiently wide distribution to have a real influence, and a list of these books should appear in the study. This very important item was omitted by Dr. Bowden.

It seems rather arbitrary to assume that all problems marked by less than 25% of the 682 ordinary parents on the basis of *remem-*

bered use should be discarded. By this same criterion a very large proportion of all the subjects we are now teaching would most certainly be discarded.

Dr. Bowden had his jury of 682 random parents vote on an abridged list of 123 problems taken from his original list of 345 problems. Apparently 25% or more marked each of 44 problems as problems that they had used, apart from all business uses. Dr. Bowden then assumed that no other problem would have been "voted in" had he sent out the full list of 345 problems. So, in this exceedingly arbitrary and unscientific fashion, he "finds that those who filled out the blanks used only 12 per cent of all the problems listed!" From this he concluded, "If we assume that the problems used in this study are typical, we are safe in saying that the schools have been teaching more than 85 per cent more arithmetic than is required for the accomplishment of the values we have described under the life situations: Buying at the Stores, Making Change, Reading, Investing One's Savings, Writing Letters, and Traveling.

Since any slight value this study may have depends on the problems selected, it is interesting to see just what kinds of problems Dr. Bowden considers "typical." The following are selected from his list. The numbers at the left are those used by Dr. Bowden in his dissertation:

12. Reduce 1 Eagle to ¼ dollars.

16. What is the value of the Peseta[1], Lire, Crown, Yen, Tael, Peso, shilling, franc and centime, in United States Money?

24. Do you use the German mark, pfennig, thaler?

33. What will be the cost of 1 T. 6 cwt. 4 lb. 2 oz. of coal @ 4c per pound?

34. What will 1 sq. mi. of land cost @ $1.00 per sq. ft.?

45. Reduce 8 lbs. 9 oz. 13 pwt. 17 grams to grams.

49. Reduce 3 yd. to barleycorns.

50. How many lines in 1 ft. 6 in.?

51. Reduce 1 span 4 in. 3 barleycorns to lines.

55. How many sheets of paper will be required for a 12 mo. book of 336 pages?

78. Add: 3 mon. 2 wk. 4 days 19 hr. 30 min. 3 sec. 1 mon. 1 wk. 5 days 13 hr. 27 min. 18 sec.

93. From—200 A. 80 sq. rd. 1 sq. yd. 3 sq. ft. 100 sq. in. take 100 A. 90 sq. rd. 9 sq. yd. 4 sq. ft. 140 sq. in.

96. From—30 cord, 27 cu. yd. 120 cu. ft. 8 cu. in. take 15 cord, 28 cu. yd. 125 cu. ft. 100 cu. in.

[1] Several problems have been combined above to save space.

105. Multiply 3 A. 252 sq. rd. 10 sq. yd. 28 sq. ft. 109 sq. in. by 39.

111. Divide 114 da. 22 hr. 45 min. 18 sec. by 54.

119. Find the least common multiple of 120, 360, 144, 720 and 72.

143. Add: 1/12, 1/13, 1/14, 1/15.

160. What will be the cost of 2 1/3 lb. of meat at 33 1/5 cents a lb.?[3]

174. Multiply $\frac{1\ 1/2}{2\ 1/3}$ of $\frac{3/7}{4\ 1/2}$ by $\frac{2/3}{3/4}$

175. Add: \$16 1/16; \$9 1/3; \$5 7/16; \$2 13/16.

208. Reduce .035 pk. to pts.

209. Reduce .005 yd. to inches.

210. Reduce .0054875 A. to sq. rd.

245. What is a man's income who owns 20 1000 dollar U. S. bonds, when gold is 107?[3]

246. What must be paid for 6% bonds to realize an income of 8%?[4]

258. The interest for 5 yr. 8 mo. 24 da. at 6% is \$28.38; what is the principal?

269. What is the difference between simple and annual interest on \$1000 for 5 yr. at 6%?

270. What is the difference between compound interest and simple interest on \$5325 for 5 years 6 months at 7% payable semi-annually?[5]

272. Find the date when due, bank discount, time of discount, and proceeds of a note of \$2400 dated Oct. 16, 1876, due Jan, 1, 1878; with interest at 8% discounted July 26, 1877 at 10%.[6]

281. 4 is the ratio of what number to 13?

282. 2 3/4 is the ratio of 23 3/8 to what number?

313. Extract the cube root of 91125.

314. Extract the cube root of 53.157376.

315. Extract the cube root of 9 1/6.

316. A cubical box contains 512 half inch cubes. What are the dimensions?

317. A cubical excavation contains 450 cu. yd. 17 cu. ft. What are the dimensions?

337. What is the 99th term of a decreasing series in which the first term is 329 and the common difference is 7/8?

338. The first term in a series is 2, the last 20, and the number of terms 7. What is the common difference?

339. The extremes in a progression are 2 and 50; the number of terms 24; find the sum of the series. Can any reader solve this problem?

[3] We do not have scales that will weigh 1/3 lb.

[3] Is this a Civil War relic? How would you solve this problem?

[4] This would puzzle the best banker or broker. When are these bonds due?

[5] Annual interest was a "hybrid" between simple and compound interest. It has been practically obsolete for fifty years or more.

[6] We may be fairly sure that this problem was taken from a book written before 1876 or 54 years ago.

340. The first term of a decreasing series is 262144; the ratio 4; the number of terms 9. Find the last term.

341. Find the sum of the infinite series 1/3, 1/9, 1/27, etc.

342. Find the sum of an infinite series of which the greatest term is .3 and the ratio 10.

Many of these problems were practically obsolete fifty years ago, others were retained in the text books but seldom or never taught. Practically all of the 345 problems on this list can be found in arithmetics published prior to 1850, while on the other hand, there are none of the modern type of problems found in all good recent texts. A person who considers these problems "typical" of what we are teaching today is certainly not qualified to conduct any curriculum studies in arithmetic. It is well that Dr. Bowden warns us, "The tools of a sociologist and his techniques are elusive and difficult of comprehension." This enables us to expect almost anything.

As a last straw we may consider the reliability of the 682 random parents who are to determine the arithmetic "Content for the Present and the Coming Generation!" The following problems are rated 3, 5, and 9 in social value:

26. If one peck of peaches sells for $1.00, what will 1 pint cost? (S.V.R. 3)[7]

162. What is the cost of 1 2/3 yd. of muslin at $3/20 per yd? (S.V.R. 5)

35. What will be the cost of 3888 sq. in. of carpet at $5.00 per sq. yd.? (S.V.R. 9)

In spite of the high rating given them, these are freak problems. We do not buy peaches by the pint, and no one ever saw 15 cents disguised as $3/20. Of course, none of these people ever had to find the cost of 3888 sq. in. or any other number of square inches of carpet at $5.00 per sq. yd. In contrast, the following problems are given the low social value ranks of 25, 29.5, and 34:

319. How many square feet in a floor 17 ft. long and 15 ft. wide? (S.V.R. 25)

32. What is the weight of 180 iron castings each weighing 75 lbs.? (S.V.R. 29.5)

202. Multiply .19 by .125. (S.V.R. 34)

121. In 3 apples, how many fourths? (S.V.R. 34)

The remarkable fact that problem 35 was placed so far ahead of problem 319 shows how little trust we can place in the jury of 682 parents. They appear no better qualified to evaluate arithmetic content than the judge.

[7] Read "Social Value Rank 3."

The kindest thing we can say about this study is that it is worthless. It is so palpably unscientific that it will not fool any person who has had any real training in education, but many editors and others without this training have accepted the conclusions of the study and have launched into tirades against arithmetics that have done considerable damage.

As Professor David Eugene Smith says in the Third Yearbook of the National Council of Teachers of Mathematics: "There are just as elaborate tables seeking to prove things in education, which tables have secured degrees for their makers and yet are so ridiculous that this writer, at least is ashamed to have his European friends see them lest they judge all American scholars by such unscientific work."

Subscribers, Please Note

The readers of THE MATHEMATICS TEACHER are no doubt interested in the News Notes that appear from time to time in the magazine. However, it has been difficult to obtain these notes without a great deal of trouble. Will all of our readers please send in any notes of interest or importance promptly so that this part of each issue of the magazine may become a vital part. Notices of meetings of mathematics teachers should be sent in prior to the time of meeting in order that advance announcements of such meetings may appear in THE TEACHER and thus notify our readers where these meetings are to be held.

—THE EDITOR

Pythagoras

Second of the Great Philosophers of Greece

ACCORDING to Professor David Eugene Smith, Pythagoras[1] is the most interesting figure in the history of ancient mathematics. He ranks him first "partly from the mystery surrounding his life, partly from his own mysticism, partly from the brotherhood which he established and partly from the unquestioned ability of the man himself."[2]

The date and place of Pythagoras' birth are both unknown and his parentage is quite uncertain. He seems to have been born between 580 and 568 B.C. of our calendar. However, we know that he lived at a time "after Greece had enjoyed two centuries of commerical activity, and at dawn of that golden age which began in Athens in the 6th century B.C. and closed for that city at the end of the 5th century B.C. Moreover, we know that he was one of the great makers of the civilization of that period. He is said to have "changed the study of geometry into the form of a liberal education."

Professor Smith says that "The fact that arithmetic and geometry took such a notable step forward at this time was due in no small measure to the introduction of Egyptian papyrus into Greece. This event occurred about 650 B.C."

Our knowledge of the life of Pythagoras is very limited although many interesting accounts have been invented by various writers These accounts relate to "his travels, his miraculous powers, and his teachings." It is said that he sought out Thales as his teacher. His tory has no record of his doings for a considerable part of his earlier life. If we judge from his teachings, he must have had contact with the Orient. "His mysticism of numbers is quite like that found earlier in Babylon, and indeed his whole philosophy savors much more of Indian than of the Greek civilization in which he was born. According to our best evidence the familiar proposition which bears his name was known in India, China, and Egypt before his time, and all that can be claimed for him in relation to it is that he may have given the earliest general demonstration of its truth."

[1] See frontispiece.

[2] See Smith, David Eugene, History of Mathematics, Vol. 1, p. 69. Ginn & Co., 1923.

Pythagoras divided his audiences into two classes: the Probationers (or listeners) and the Pythagoreans (or mathematicians). After three years in the first class a listener could be initiated into the second class, to whom were confided the main discoveries of the school.

The Pythagoreans formed a brotherhood in which each member was bound by oath not to reveal the teachings or secrets of the school. Their food was simple, their discipline severe, and their mode of life arranged to encourage self-command, temperance, purity, and obedience.

The triple interwoven triangle, or pentagram (star-shaped regular pentagon), was used as a sign of recognition, and was to them a symbol of health. It is related that a Pythagorean while traveling fell ill and, although carefully nursed by a kind-hearted innkeeper, was unable to survive. Before dying, however, he inscribed the pentagram star on a board and begged his host to hang it up outside. This the host did; and after a considerable length of time another Pythagorean, passing by, noticed the sign and, after hearing the innkeeper's story, rewarded him handsomely. One motto of the brotherhood was: "A figure and a step forwards; not a figure to gain three oboli."

The views of society advocated by the brotherhood were opposite to those of the democratic party of Pythagoras's time, and hence most of the brotherhood were aristocrats. For a short time the Pythagoreans succeeded in dominating affairs, but a popular revolt in 501 B.C. led to the murder of many prominent members of the school, and Pythagoras himself was killed shortly afterwards.

Though the brotherhood no longer existed as a political party, the Pythagoreans continued to exist a long time as a philosophical and mathematical society, but to the end remained a secret organization, publishing nothing, and thus leaving us little information as to the details of their history.

Using Ratio in Problem Solving

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THE WORD ratio is the name for a simple and extremely useful idea. The traditional discussion of ratio and proportion would not be likely to convince anyone that the idea was either very simple or very useful. The fact that this idea is so often misused is evidence that we must teach our pupils to understand ratios better and use them more advantageously. Ratio and proportion still maintain their conventional co-partnership in most of our arithmetics, usually setting forth an array of problems inherited from some generations past. The introduction of the ratio concept in the elementary grades, however, tends to break up the partnership and show ratio as an important subject.

The first idea about a ratio for the pupil to get is that a ratio is a number. The ratio of the circumference of a circle to the diameter is 3.14, approximately. It is not necessary to say 3.14 to 1. We say merely that the ratio is 3.14. Ratios used as multipliers are very common in applied mathematics where they are called ratios, coefficients, factors, and other names. The ratio idea and that of the proportionality of numbers underlies a considerable part of the material used in arithmetic.

When two numbers are so related that the first is three times the second, they are said to be in the ratio of 3 to 1. If the first is $1\frac{1}{2}$ times the second, they are in the ratio of $1\frac{1}{2}$ to 1 or 3 to 2. Here we have the second idea about ratio. A ratio tells how many times. This simple idea must be firmly fixed in the pupil's mind.

Next the pupil should discover that in order to find a ratio he divides. He should know, of course, that a ratio may be expressed either as a common fraction or in decimal form. The following principle should be considered at this point.

PRINCIPLE: *To find the ratio of one quantity to another, express both in the same unit of measure before dividing.*

Thus: (a) The ratio of 2 lb. to 8 oz. is 32 oz. to 8 oz.

(b) The ratio of 3 doz. oranges to 2 oranges is the ratio of 36 oranges to 2 oranges.

The third idea about ratios involves problems solving. For example, in the problem: "Mr. White earned $27 in 6 days. How much will he earn in 14 days at the same rate?" The pupil should be taught to form a ratio expressing how many times as much. The ratio is either $\frac{6}{14}$ or $\frac{14}{6}$. If the answer is to be less than $27 the result will be $\frac{6}{14}$ of $27. If the answer is to be more than $27 the result will be $\frac{14}{6}$ of $27. The pupil's judgment will tell him which ratio to use. Only the simplest phases of proportion are needed in arithmetic, and they may well be treated informally and untechnically. The third idea about ratios, as given above, is all that is needed for the solution of problems involving the proportionality of numbers occurring in arithmetic. Therefore, it seems unnecessary to make a marked distinction between ratio problems and proportion problems. It is quite proper to treat proportionality as a ratio idea.

The following examples illustrate two types of ratio problems.

(1) If George can ride a bicycle 40 feet while Frank runs 30 feet, how far can George ride while Frank runs 45 feet?

(2) If Fred can ride a bicycle 60 feet while Jack runs 50 feet, how far can Jack run while Fred rides 70 feet?

(1) How long should it take 10 men to do some work that 15 men can do in 20 days?

(2) If 9 men can complete a job in 32 days, how many men would be required to complete the job in 24 days?

The two types should be presented simultaneously so that the pupil may have an opportunity to compare and analyze his method. He should, consequently, use judgment in planning the solution. After a little practice the pupil will learn to recognize ratio problems so that he can say to himself, "That's a ratio problem."

The study of a widely used intelligence test reveals the fact that 10 per cent of the questions asked involve a knowledge of ratio. The author evidently believes that the reasoning involved in ratio problems is very similar to the reasoning involved in other situations.

The satisfactory attainment of the teaching of arithmetic requires an understanding of the ratio ideas and the situations in which they are applied. A sufficient amount of practice and drill is essential to make the applications quickly and accurately. The ratio idea is so simple and useful that it should be introduced as soon as thought problems are studied. Here is an excellent opportunity to teach children the logical steps in the process of reasoning.

Commercial Mathematics in a Small High School

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I. Need For Change in Present Requirements

Most references on this subject express the view that arithmetic is the only mathematical study necessary for the pupil preparing to enter the business world, and such is the practice at the present time. Mathematics for the commercial student is confined to a one-year course in commercial arithmetic. However from personal experience as a statistician, I have reached the conclusion that something is radically wrong. Either commercial students are not receiving enough mathematical training; the subject matter taught is not of value to them after leaving school; or the material is not taught, so that they acquire the ability to use it or rather apply it to the new conditions which they must meet. I am inclined to believe that the fault does not lie in any one of the above conditions, but in all of them. There should be a radical revision of both our curricula and teaching methods, and a realization of the need for more thorough mathematical preparation for the commercial student.

Probably by recalling some of the incidents which have come to our attention personally, we shall be able to illustrate the need for a change in our mathematical preparation. We have continually encountered difficulties when trying to assume that the office girls, whose work we have supervised, have some mathematical knowledge. Our experience has been confined to large corporations, where the work of each individual is necessarily specialized. In small concerns the need for proper mathematical preparation is still greater.

We once handed a statement to a stenographer to be typed, before completing all of the work, because of the necessity for beginning a "rush job," requesting her to type the column totals as well. The statement contained several columns of figures, similar to the following:

Shipyard, Hull No., Tonnage Completed, Per cent of Completion, To be Constructed. When we proofread the typed statement we found

that for each shipyard the hull numbers of the various ships and the percentages of completion had been totalled as well as the tonnage. Figures had absolutely no meaning for that girl.

A clerk once told us that she had never had any use for algebra after graduating from school. Yet when her chief asked her a question in a conference that same day she could not answer it. He knew that 54 per cent of the sum of two numbers and 61 per cent gave 58 per cent house service expense, and from that fact wanted to find the combined percentage when 87 and 41 per cent were given as corresponding figures by another telephone company. The business girl does not meet definite problems found in textbooks, but she is constantly confronted with conditions which require the ability to apply principles and rules learned. But in order to use these rules intelligently, they must be learned in that way, not mechanically.

In work with which we were recently connected all statistical statements had to be checked before they could be sent from the office. Routine work was checked by the comptometer operators. Frequently however estimates and statistics had to be rushed to the executive offices, so that there was not time for checking by machine. Usually we were compelled to stop other work and use the slide rule, merely because no one else knew how to use this instrument.

Speed and accuracy are the two most important essentials in any business office. Statistics are demanded for use in conferences almost immediately. Practice in estimating results roughly, so as to eliminate ridiculous errors, is necessary. By so doing, the commonest error in operating an adding machine or comptometer, that of an error in decimal point, would be eliminated. We have received supposedly checked reports in which the product of 25 by 416.35 was given as the product of 25 by 41.635. Upon requesting a clerk to perform some computations as quickly as possible, she confessed that she could multiply by means of the comptometer much more speedily than she could perform the operation of division. We thereupon supplied her with a reciprocal table, so that she could use multiplication instead of division on the machine; however she could not use the table of reciprocals, and we were compelled to give her the equivalent figures before she could proceed.

We could continue to enumerate incidents which would show conclusively that the present training which the commercial student receives in high school is not fitting him for his job. He may be able

to solve a compound interest or foreign exchange problem, but when will he be required to do so? The ability to take a series of figures and represent them graphically so as to show a comparison or to indicate a certain tendency, an increase in costs or a decrease in sales, will be of infinitely greater value to him when he enters an office. Our education aim should be to prepare him for his future lifework as thoroughly and as satisfactorily as possible.

II. Method of Teaching

In discussing a revision of methods of teaching, several factors enter. The textbook is followed too closely by the average teacher. Typical exercises involving income tax, partnership or discount problems are considered because chapters are devoted to them in the textbook. Too frequently no effort is made to correlate problems under a class project. Secondly, the subject matter must be taught in such a way that the student will be able to apply principles learned to other slightly different problems. Mere mechanical application of rules is not conducive to the intelligent use of these rules and principles later. The subject must be taught intelligently in order to achieve the proper results.

Furthermore the teacher of the commercial student should make the mathematics studied practical. By this term we do not mean that he should use unreal situations. No high school student will believe a mathematics teacher who states that algebra should be used as a check in order to see if one has obtained the correct change after paying for a purchase. Yet, when I was a high school freshman, I heard that statement made by a teacher of algebra. The teacher of mathematics should make his pupils realize that graphical representation is more satisfactory than columns of figures, that results can be estimated approximately, that the slide rule is an instrument whose use should be known, that logarithms may be used to simplify many business calculations.

III. Curricula

Not only should methods of teaching be improved, but there should be a change in the present curricula as well. In the larger township high schools where several courses of study are offered, the mathematics included in the commercial course is a one-year course in business arithmetic. In the smaller rural high schools where there is no

differentiation of courses, the commercial student suffers to a greater extent. Several years ago I taught in the high school of a small town of about 3,000 population. The school had an enrollment of approximately 170 students. There were 6 classes in mathematics. The principal of the school insisted that the traditional mathematics courses be given in elementary and intermediate algebra, plane and solid geometry, and trigonometry, so that any student who took the College Entrance examinations in mathematics would pass these examinations. His argument was that the majority of the seniors took the examinations of the College Entrance Board. They did; but he neglected to take into consideration the larger number of students who dropped by the wayside, because of their inability to master the mathematics given.

Another class which is neglected is that consisting of students who graduate from the small high school and then enter a business school in the state capital. Such a course is necessary if they desire to obtain a clerical or stenographic position in the town bank, court house, or electric company. They have not obtained in high school the type of mathematical training necessary in business, so that they are compelled to obtain further preparation in essentially high school subjects. Consider the time which could be saved, if commercial applications were introduced into the mathematics courses studied in high school.

Must the best interests of these two classes of students be sacrificed for those of the comparatively few who continue their studies in college? Differentiation of courses is naturally impossible in a school of this size, but the mathematics courses should be so modified that a greater number may be benefited. Does the course in generalized mathematics satisfy this need? I have reached the conclusion that selected mathematical topics rather than general mathematics is the best preparation for the commercial student.

Adverse criticism is worth nothing, unless such criticism leads to constructive measures. What is the remedy? How may the small high school give to the students of a town the mathematics which is most essential to their future vocational interests? The one year of commercial arithmetic or the traditional course of one year of algebra followed by one year of geometry should be replaced by a one-year course in correlated mathematics, including selected topics from arithmetic, algebra and geometry. Of the topics usually considered in the arithmetic course a review of the use of fractions and decimals,

rapid drill in the fundamental operations, aliquot parts, percentage, discount, simple interest, building and loan association problems, insurance, taxes, and stocks and bonds should be discussed. The algebraic topics should include a thorough discussion of statistical graphs, their construction and interpretation; formulas, their construction, evaluation and solution; dependence; checks and approximate results; the use of logarithms and the slide rule; the algebraic solution of problems, involving linear equations. Some teachers may protest that the subject of logarithms is too difficult for ninth year pupils. However the criticism is not justified, since the topic has been taught successfully to the ninth year class of a New York City high school for years. The following geometric topics should be considered: simple geometric constructions, drawing to scale, the logical proof of originals based on congruent and similar triangles. Familiarity with a geometric proof, the selection of what is given and what must be proved, the logical arrangement of the various steps in a proof, is of great value. Training in geometric reasoning may be transferred to the successful outlining of a plan for producing a neat, carefully arranged, typewritten sheet of statistics. It is essential that commercial students receive some training in algebraic and geometrical methods of solving problems, in addition to those of arithmetic.

To Postulate or not to Postulate

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HOW MANY of the first principles and theorems of the traditional course in Plane Geometry shall be postulated? On one side are those who say that we should scrap all of the demonstrative work and confine our attention to intuitive geometry and to practical applications.

An example of this type of literature is a pamphlet of 58 pages entitled, *Essentials of Algebra and Geometry* by Frank Millet Morgan.[1] The first sentence of the introduction is, "The following lessons are intended to give you the algebra and geometry needed for your subsequent study of trigonometry." This book was intended for adults who were looking for shortcuts but claims to give the essentials for future work in mathematics. In this pamphlet all geometry is postulated. There is only one theorem for which any approximation of formal proof is given, and that is the algebraic proof for the relationship between the exterior angle and the opposite interior angles of a triangle. Only 16 of the 58 pages are devoted to geometry in any form.

If then the only purpose of geometry is to teach the construction of a few figures, the use of the properties of similar figures and circles, and the practical application of the Pythagorean theorem, then let it be done in two or three weeks and devote the remaining time to a course in design for linoleum, figured silks, and modernistic furniture.

No one who has ever really glimpsed the beauty of geometry, will for one instant agree with the above suggestion. It is not necessary to point out to the readers of THE MATHEMATICS TEACHER the reasons for teaching geometry. Sister Alice Irene has most ably shown in the December, 1929 issue how one phase of the subject can be taught to girls.

If we are agreed that we teach geometry for many reasons other than for the practical applications, how many of the fundamental facts shall we postulate? Euclid's Common Notion is, "Things which coincide with one another are equal to one another." In reference to this, Heath[2] says,

[1] See Bibliography at the end of this discussion and similarly for the other references.

It seems clear that the Common Notion, as here formulated, is intended to assert that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal. . . . .

Concerning the idea of superposition philosophers and geometers have argued for centuries. To the class room teacher who is in daily contact with the young mind, it would seem that Bertrand Russell[3] gives the best argument for or evasion of superposition, call it whichever you wish, in the following words:

The apparent use of motion here is deceptive; what in geometry is called a motion is merely the transference of our attention from one figure to another. Actual superposition which is nominally employed by Euclid, is not required; all that is required is the transference of our attention from the original figure to a new one defined by the position of some of its elements and by certain properties which it shares with the original figure.

Heath[4] gives a fine résumé of the arguments for and against Common Notion. H. C. Christofferson[5] says,

Hilbert exposed the fallacy of the superposition proof, and progressive mathematicians no longer attempt to postulate superposition so as to prove the two congruence theorems when they can as well postulate the congruence theorems themselves, with no loss of rigor, and in fact with considerable gain.

How many secondary school teachers of geometry are willing to follow the above suggestion? If all "progressive mathematicians" are agreed as to this, then the College Entrance Examination Board should remove the three congruence theorems from the starred list. If we are to act upon this suggestion, suppose we go one step further and postulate the fact that the angles opposite the equal sides of an isosceles triangle are equal. Any high school pupil will intuitively accept this fact as readily as that of the congruence of triangles.

If we are not ready for Professor Christofferson's suggestion, then let us begin our demonstrative geometry, as some of our recent textbooks[6] do, with the two congruence theorems which depend upon superposition for their proof. Place the "pons asinorum"[7] as number three and prove it by Euclid's own method. Now hear the protests: "Too hard!" "Too long and involved!" "Entirely out of place in a modern text!" "Has not been used for fifty years!" Some of these statements are true. If it is too hard and too long, then let us ask our pupils to accept the fact for two or three weeks, and then go back for the proof. The difficulties will have disappeared, and some of the brighter pupils will have already mastered it and will ask why it was not taken in order.

This proof is a fine illustration of logical reasoning which shows how entirely dependent one theorem is upon previous facts either proved or postulated. Above all else, it frees us from the deadly sin of reasoning in a circle. The average student in geometry never detects the lack of logical procedure and accepts without a word the postulates, that *an angle can be bisected,* or that *at a given point in a given line an angle can be constructed equal to a given angle.* He later proves these facts and fails to connect them with the previous use made of them. But now and then some keen student discovers the trick and is quick to ask why the textbook is not consistent. The only answer is that it has been thus presented to make it easy for him. He resents this and wonders how many more times he is to be imposed upon. When this situation arises, it gives an opportunity to go back to Euclid's proof.

Even Legendre in the preface to the 11th edition of *Elements of Geometry*[8] says,

> The method of the ancients is very generally regarded as the most satisfactory and the most proper for representing geometrical proof. It not only accustoms the student to great strictness in reasoning, which is a precious advantage, but it offers at the same time a discipline of peculiar kind. . . . .

However, when he comes to prove that the angles opposite the equal sides of an isosceles triangle are equal, he forgets about the "strictness in reasoning" and proceeds in a circle.

If it is necessary to shorten the course in Plane Geometry, as is the modern trend, let us do it by omitting theorems and not by postulating more of the fundamental facts. Above all, let us free our textbooks of the absurdity of reasoning in a circle in a subject which is supposed to show the necessity of rigor in mathematical reasoning.

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(1) At the time of publication Prof. Morgan was assistant Professor of Mathematics at Dartmouth College. (1919)
(2) Volume 1 of *The Thirteen Books of Euclid's Elements,* page 225.
(3) Ibid., page 227.
(4) Ibid., pages 224-228.
(5) THE MATHEMATICS TEACHER, December, 1928.
(6) *Modern Plane Geometry,* Clark-Otis, and *Plane Geometry,* Betz and Webb, are two such textbooks.
(7) Heath, Volume 1, page 251.
(8) *Elements of Geometry,* Legendre, published by Hillard and Metcalf, Cambridge, Massachusetts, 1819.

NEW BOOKS

The Teaching of General Mathematics in the Secondary Schools of the United States. Teachers College, Columbia University Contributions to Education No. 386. By Clarence McCormick. Bureau of Publications, Teachers College, Columbia University, New York. 1929. Pp. 173. $1.75.

This book records the history, success, and extent of teaching general mathematics in American high schools. In terms of the different chapters of the book, we might say that the study attempts to do the following things:

1. To distinguish between the different types of mathematics.
2. To give the history of the general mathematics movement.
3. To record the studies made in regard to the success of the teaching of general mathematics in the high school.
4. To determine the type of mathematics in the seventh, eighth, ninth, and tenth grade textbooks for different periods of time. In this way the present trends may be determined.
5. To examine courses of study for the same four grades in order to ascertain the extent of the teaching of mathematics as indicated by these courses.
6. To determine as far as possible the opinion of high school teachers in regard to the relative merits of general and traditional mathematics.
7. To determine the opinions of specialists in the methods of teaching mathematics in regard to the merits of the two types.
8. To arrive at certain conclusions and to make certain suggestions concerning the teaching of high school mathematics.

In attacking this list of problems, the author obtained his data from published articles pertaining to general mathematics, records of the University of Minnesota students who had studied general mathematics in high school, questionnaires to teachers of mathematics, textbooks in the seventh, eighth, ninth, and tenth grades, printed courses of study, and personal requests to a few specialists in the teaching of mathematics.

This procedure resulted in finding out the following things:

1. There is much confusion as to the meaning of general mathematics.
2. The present general mathematics movement grew out of the movement for correlated mathematics, which emphasized the fusion of the different branches.
3. Data collected show that in two schools, the University of Minnesota and the University of Chicago, students who had studied general mathematics in high school for two years made slightly better marks in college mathematics than those who had studied the traditional type of mathematics for two years.
4. The percentage of students who enroll in eleventh and twelfth grade or freshman college mathematics is

about the same for both classes of students mentioned above.

5. The examination of the textbooks revealed the following facts: (1) junior high school textbooks are nearly all of the general mathematics type, (2) the seventh and eighth grade arithmetic books are gradually becoming more general in character, (3) the movement for general mathematics has brought about considerable change in ninth grade textbooks, (4) the tenth grade books have changed least of all, (5) a considerable number of general mathematics textbooks have been published for freshman college work.
6. The courses of study are gradually becoming more general.
7. There is no consensus of opinion among high school teachers as to which is the better type of mathematics.
8. The opinions of specialists in the teaching of mathematics is quite strongly in favor of some type of general mathematics.
9. The tendency at present is not to stress correlation, but it should be introduced wherever the correlation is natural.
10. There should not be two types of mathematics for grades seven and eight, one type for those pupils in the eight-year elementary school and another type for those pupils in the junior high school. Both have the same right to a worth while course in mathematics.

It is evident that the study is complete and well done. It gives us the present status of general mathematics in this country. It is not an argument for either side, though it is easy to see that the author believes in the value of teaching general mathematics. The results do not always favor him however. In some cases he has made interpretations which may not seem warranted from the data presented, but in these cases he is careful to say that he is expressing a personal opinion in stating that certain other things had probably affected the result under discussion.

It would be safe to say that no one who is interested in the progress of the teaching of mathematics can afford to be ignorant of the results of this study. These results are in no way startling, but they will show us where we stand and will give evidence for many of the opinions that have been held by progressive teachers of mathematics.

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